Lecture – 35.

Green's Theorem

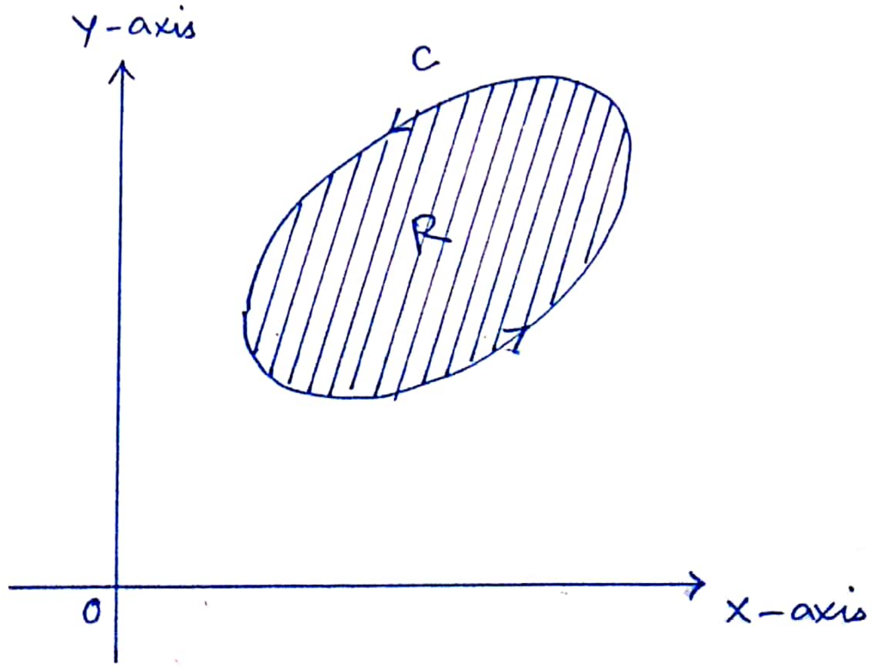
Let C be a piecewise smooth simple closed curve bounding a region R. If $f, g, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}$ are continuous on R, then

$$\boxed{\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy}$$

the integration being carried in the positive direction of C.

Note 1. Green's Thm. provides a relationship between a double integral over region R and the line integral over the closed curve C bounding R.

2. Green's Thm. is also called as First fundamental theorem of integral vector calculus.



Use Green's Thm. to evaluate : (II).

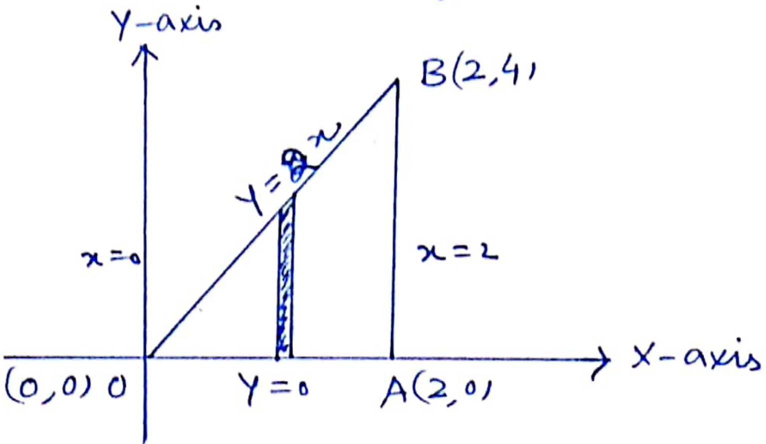
1. $\oint_C (x+y)\,dx + x^2 dy$, C is the triangle with vertices (0,0), (2,0) and (2,4), taken in that order.

Sol: Here $\oint_C (x+y)\,dx + x^2 dy$

Comparing it with: $\oint_C f\,dx + g\,dy$

$f = (x+y)$ | $g = x^2$

$\Rightarrow \frac{\partial f}{\partial y} = 1$ | $\frac{\partial g}{\partial x} = 2x$



In given figure
$$R: \left\{ \begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le 2x \end{array} \right\}$$

By Green's Thm:

$$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$\Rightarrow \oint_C (x+y)\,dx + x^2 dy = \int_{x=0}^{2} \left[\int_{y=0}^{2x} (2x-1)\,dy\right] dx$$

$$= \int_{x=0}^{2} \left[2xy - y\right]_{y=0}^{2x} dx$$

(III)

$$\Rightarrow \oint_C (x+y)\,dx + x^2\,dy = \int_{x=0}^{2} \left[2xy - y\right]_{y=0}^{2x} dx$$

$$= \int_{x=0}^{2} \left[\left[2x(2x) - 2x\right] - 0\right] dx$$

$$= \int_{x=0}^{2} (4x^2 - 2x)\,dx$$

$$= \left[4\frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}}\right]_{x=0}^{2}$$

$$= \left[4\left(\frac{8}{3}\right) - (2)^2\right] - 0$$

$$= \frac{32}{3} - 4$$

$$= \frac{32-12}{3} = \frac{20}{3}$$

$$\Rightarrow \boxed{\oint_C (x+y)\,dx + x^2\,dy = \frac{20}{3}}$$ Ans.

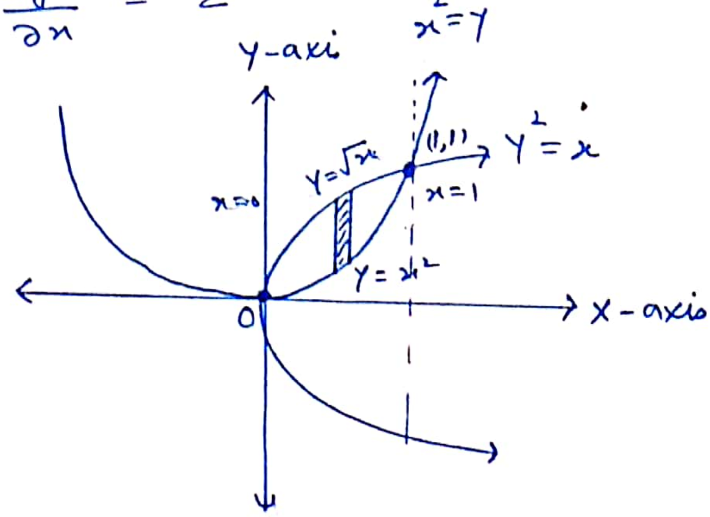
2. $\oint_C (x^2+y^2)\,dx + (y+2x)\,dy$ where C is boundary of region in 1st quadrant bounded by $y^2 = x$; $x^2 = y$

Sol: Here $\oint_C (x^2+y^2)\,dx + (y+2x)\,dy$

Comparing with: $\oint_C f\,dx + g\,dy$

$$f = (x^2+y^2) \qquad \Big| \qquad g = (y+2x)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 2y \qquad \Big| \qquad \frac{\partial g}{\partial x} = 2$$



In given figure:

$$R: \left\{ \begin{array}{c} 0 \le x \le 1 \\ x^2 \le y \le \sqrt{x} \end{array} \right\}$$

By Green's Thm:

$$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$\Rightarrow \oint_C (x^2+y^2)\,dx + (y+2x)\,dy = \int_{x=0}^{1}\left[\int_{y=x^2}^{\sqrt{x}} (2-2y)\,dy\right] dx$$

$$= \int_{x=0}^{1} \left[2y - \frac{2y^2}{2}\right]_{y=x^2}^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1} \left[(2\sqrt{x} - x) - (2x^2 - x^4)\right] dx$$

$$= \left(\frac{4}{3} - \frac{1}{2}\right) - \left(\frac{2}{3} - \frac{1}{5}\right) = \frac{11}{30} \quad \text{Ans}$$

$$\Rightarrow \boxed{I = \frac{11}{30}} \quad \text{Ans.}$$

(I).

3. $\oint_C (x^3 dy - y^3 dx)$, C is circle $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$

Sol: Here $\oint_C (x^3 dy - y^3 dx)$

comparing with $\oint_C f\,dx + g\,dy$

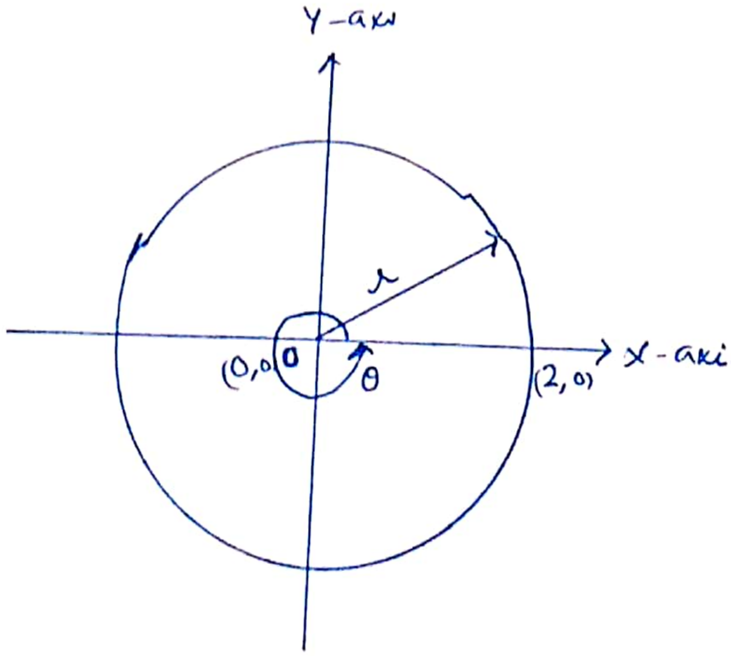
$f = -y^3 \Rightarrow \frac{\partial f}{\partial y} = -3y^2$ | $g = x^3$, $\frac{\partial g}{\partial x} = 3x^2$

By Green's Thm:

$$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$\Rightarrow \oint_C (x^3 dy - y^3 dx) = \iint_R (3x^2 + 3y^2)\,dx\,dy$$

$$\Rightarrow \oint_C x^3 dy - y^3 dx = 3\iint_R (x^2+y^2)\,dx\,dy \quad \text{——— (1).}$$



In Polar co-ordinates.

$x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow x^2 + y^2 = r^2$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$dx\,dy = r\,dr\,d\theta$

$\therefore$ From eqn: (1).

$$\oint_C x^3 dy - y^3 dx = 3\int_{r=0}^{2}\int_{\theta=0}^{2\pi} (r^2)\cdot r\,dr\,d\theta$$

$$= 3\int_{r=0}^{2} r^3 dr \int_{\theta=0}^{2\pi} d\theta$$

$$= 3\left[\frac{r^4}{4}\right]_{r=0}^{2}\left[\theta\right]_{\theta=0}^{2\pi}$$

$$= 3\left[\frac{16}{4} - 0\right]\left[2\pi - 0\right]$$

$$= 3 \times 4 \times 2\pi$$

$$= 24\pi$$

$$\Rightarrow \boxed{\oint_C x^3 dy - y^3 dx = 24\pi} \quad \text{Ans.}$$

* **Imp. Result:**

Let C be a +vely oriented simple closed path enclosing a simply connected region R, then by Green's Thm:

$$\text{Area of region, } R = \oint_C x\,dy = -\oint y\,dx$$

$$= \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

Line Integral Independent of Path

Let C be a curve in a simply connected domain D.

Then 1. $\int_C (f\,dx + g\,dy)$ is independent of path C

if and only if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

||ly 2. $\int_C (f\,dx + g\,dy + h\,dz)$ is independent of

path C iff $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$, $\frac{\partial h}{\partial x} = \frac{\partial f}{\partial z}$

1. Show that the line integral:

$$\int_P^Q 2xy^2\,dx + (2x^2y+1)\,dy$$ is independent of path

of integration.

Sol: $$\int_P^Q 2xy^2\,dx + (2x^2y+1)\,dx$$

Comparing with: $$\int_P^Q f\,dx + g\,dy$$

$f = 2xy^2$	$g = (2x^2y+1)$
$\Rightarrow \frac{\partial f}{\partial y} = 4xy$	$\frac{\partial g}{\partial x} = 4xy$

Since $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

So, Given line integral is independent of path.

2. $\int_P^Q (3x^2 + 2xyz)\,dx + (1 + x^2 z)\,dy + x^2 y\,dz$

Sol: Comparing with:

$$\int_P^Q f\,dx + g\,dy + h\,dz$$

Here $f = (3x^2 + 2xyz)$ | $g = (1 + x^2 z)$ | $h = x^2 y$

$\frac{\partial f}{\partial y} = 2xz$ | $\frac{\partial g}{\partial x} = 2xz$ | $\frac{\partial h}{\partial x} = 2xy$

$\frac{\partial f}{\partial z} = 2xy$ | $\frac{\partial g}{\partial z} = x^2$ | $\frac{\partial h}{\partial y} = x^2$

Since
$$\left.\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial z} &= \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial x} &= \frac{\partial f}{\partial z} \end{aligned}\right\}$$

So, Given line integral is independent of path.